Lecture 5 - Objective and Outcomes

We look at applications of our work on multivariate distributions.

- Laws of large numbers and central limit theorem
- Transformations of random variables
- Multivariate normal distribution
- (Random sums)

After reviewing the notes you should:

- be able to prove the central limit theorem,
- know how to work out the density of a transformed random variables,
- understand how correlation is represented in multivariate normal distributions.

Types of convergence

Let $Y, Y_1, Y_2, \ldots,$ be a collection of random variables.

1. Convergence in distribution: $Y_n \xrightarrow{D} Y$ if $P(Y_n \leq y) \longrightarrow P(Y \leq y)$ as $n \longrightarrow \infty$.

2. Convergence in probability: $Y_n \xrightarrow{P} Y$ if $P(|Y_n - Y| > \varepsilon) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $\varepsilon > 0$.

3. Convergence almost surely: $Y_n \xrightarrow{a.s.} Y$ if $A = \{\omega \in \Omega : Y_n(\omega) \longrightarrow Y(\omega)$ as $n \longrightarrow \infty\} \Rightarrow P(A) = 1$.

Laws of large numbers

1. The law of large numbers:
   $\frac{1}{n}S_n \xrightarrow{D} \mu$ as $n \longrightarrow \infty$.

2. Weak law of large numbers:
   $\frac{1}{n}S_n \xrightarrow{P} \mu$ as $n \longrightarrow \infty$.

3. Strong law of large numbers:
$\frac{1}{n}S_n \xrightarrow{a.s.} \mu$ as $n \longrightarrow \infty$.

4. Central limit theorem:

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{D} N(0,1) \quad \text{as} \quad n \longrightarrow \infty.$$

Bivariate transformations

Suppose continuous random variables (X_1, X_2) transformed to (Y_1, Y_2) by $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

The joint density of Y_1 and Y_2 is given by

$$f_{Y_1,Y_2}(y_1, y_2)$$
$$= \begin{cases} f_{X_1,X_2}(x_1(y_1,y_2), x_2(y_1,y_2))|J(y_1,y_2)| & \text{if } (y_1,y_2) \text{ is in the range of } \\ 0 & \text{otherwise,} \end{cases}$$

where $(x_1(y_1,y_2),x_2(y_1,y_2))' = T^{-1}(y_1,y_2)'$ and

$$J(y_1,y_2) = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \frac{\partial x_1}{\partial y_1}\frac{\partial x_2}{\partial y_2} - \frac{\partial x_2}{\partial y_1}\frac{\partial x_1}{\partial y_2}.$$

Standard bivariate normal

if U and V are independent standard normal random variables, and

$$X = U$$
$$Y = \rho U + \sqrt{1-\rho^2}V$$

then $Y \sim N(0,1)$ and $\mathrm{Corr}(X,Y) = \rho$. The joint distribution is a standard bivariate normal;

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-(x^2-2\rho xy+y^2)/(2(1-\rho^2))}.$$

General bivariate normal

if $X^* = \mu_X + \sigma_X X$ and $Y^* = \mu_Y + \sigma_Y Y$ then $X^* \sim N(\mu_X, \sigma_X^2)$ and $Y^* \sim N(\mu_Y, \sigma_Y^2)$ with $\mathrm{Corr}(X^*, Y^*) = \rho$ and

$$f_{X^*,Y^*}(x, y) = \frac{1}{\sigma_X \sigma_Y} f_{X,Y}\left(\frac{x - \mu_X}{\sigma_X}, \frac{y - \mu_Y}{\sigma_Y}\right).$$

Conditional distribution

of Y^* given X^* is

$$Y^*|X^* = x \;\sim\; N(\mu_y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X),\; \sigma^2_Y(1 - \rho^2)).$$

Random sums

suppose that $X_1, X_2, \ldots$ is a sequence of independent identically distributed random variables and $S = X_1 + \ldots + X_N$ where N is a random variable.

1. Conditional

   expectation: $E(S|N) = NE(X)$,

   variance: $\mathrm{var}(S|N) = N\,\mathrm{var}(X)$,

   MGF: $M_{S|N}(t|N) = [M_X(t)]^N$.

2. Marginal

expectation: $E(S) = E(N)E(X)$,

variance: $\mathrm{var}(S) = E(N)\,\mathrm{var}(X) + \mathrm{var}(N)[E(X)]^2$,

MGF: $M_S(t) = M_N(\log M_X(t))$,

CGF: $K_S(t) = K_N(K_X(t))$.